Group Representation Theory on Mixing Times

Stijn Hanson (York)

March 10, 2014

Definition

Let G be a group and V be a complex vector space. Then $\rho : G \to \mathrm{GL}(V)$ is a representation if it is a homomorphism. Denote a representation by the pair (ρ, V).
A representation is irreducible if there are no non-trivial ρ-invariant subspaces of V. I.E. for all U such that $0 \neq U \subsetneq V$. Write the set of all equivalence classes (under some relation) of irreducible representations of G as $\hat{G}$.

Definition

Let G be a group and V be a complex vector space. Then $\rho : G \to \mathrm{GL}(V)$ is a representation if it is a homomorphism. Denote a representation by the pair (ρ, V).
A representation is irreducible if there are no non-trivial ρ-invariant subspaces of V. I.E. for all U such that $0 \neq U \subsetneq V$. Write the set of all equivalence classes (under some relation) of irreducible representations of G as $\hat{G}$.

Definition

Let p be some probability measure on a group G. We define the Fourier transform of p at the representation (ρ, V) by

$$\hat{p}(\rho) = \sum_{g \in G} p(g)\rho(g)$$

Theorem (Fourier Inversion Theorem)

Let p be a probability measure on a group G and $\hat{p}$ be its Fourier transform. Then

$$p(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \mathrm{tr}(\hat{p}(\rho)\rho(g^{-1}))$$

Theorem (Fourier Inversion Theorem)

Let p be a probability measure on a group G and $\hat{p}$ be its Fourier transform. Then

$$p(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \operatorname{tr}(\hat{p}(\rho)\rho(g^{-1}))$$

Theorem (Plancherel's Theorem)

Let p and q be some probability measures on a group G. Then

$$\sum_{g \in G} p(g^{-1})q(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \operatorname{tr}(\hat{p}(\rho)\hat{q}(\rho))$$

We can show that the matrix associated to any representation (ρ, V) is unitary. Thus

$$\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*$$

where * represents the adjoint.

We can show that the matrix associated to any representation (ρ, V) is unitary. Thus

$$\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*$$

where * represents the adjoint. By taking $q(g) = p(g^{-1})^*$ we get

Theorem (Plancherel (kinda))

$$\sum_{g \in G} |p(g)|^2 = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \mathrm{tr}(\hat{p}(\rho)\hat{p}(\rho^*))$$

Theorem

Let p be some probability measure on a finite group G and u be the uniform measure. Then

$$|G| \sum_{g \in G} |p^{(t)}(g) - u(g)|^2 = \sum_{\rho \in \hat{G}}^{*} d_\rho \mathrm{tr}(\hat{p}(\rho)^{(t)} (\hat{p}(\rho)^{(t)})^*)$$

where the sum is over all non-trivial irreducible representations.

We can use this to bound total variation distance with the simple observations that

$$\|p^{(t)}(x) - u\|_{\mathrm{TV}}^2 \leq \frac{|G|}{4} \sum_{g \in G} |p^{(t)}(g) - u(g)|^2$$

$$\|p^{(t)}(x) - u\|_{\mathrm{TV}}^2 \geq \frac{1}{4} \sum_{g \in G} |p^{(t)}(g) - u(g)|^2$$

by the Cauchy-Schwarz inequality and Pythagoras' theorem respectively.

We can use this to bound total variation distance with the simple observations that

$$\|p^{(t)}(x) - u\|_{\mathrm{TV}}^2 \leq \frac{|G|}{4} \sum_{g \in G} |p^{(t)}(g) - u(g)|^2$$

$$\|p^{(t)}(x) - u\|_{\mathrm{TV}}^2 \geq \frac{1}{4} \sum_{g \in G} |p^{(t)}(g) - u(g)|^2$$

by the Cauchy-Schwarz inequality and Pythagoras' theorem respectively. These give

$$\frac{1}{4|G|} \sum_{\rho \in \hat{G}}^{*} d_\rho \mathrm{tr}(\hat{p}(\rho)^{(t)}(\hat{p}(\rho)^{(t)})^*) \leq \|p^{(t)} - u\|_{\mathrm{TV}}^2 \leq \frac{1}{4} \sum_{\rho \in \hat{G}}^{*} d_\rho \mathrm{tr}(\hat{p}(\rho)^{(t)}(\hat{p}(\rho)^{(t)})^*)$$

or, slightly more vaguely,

$$\|p^{(t)} - u\|_{\mathrm{TV}}^2 \asymp \sum_{\rho \in \hat{G}}^{*} d_\rho \mathrm{tr}(\hat{p}(\rho)^{(t)}(\hat{p}(\rho)^{(t)})^*)$$

Definition

Suppose that G is a group, V is a finite-dimensional vector space and (ρ, V) is a representation of G. For each $g \in G$ consider the matrix representation of $\rho(g)$, denoted $\overline{\rho(g)}$, relative to some fixed basis in V. Define a function $\chi : G \to \mathbb{C}$ by $\chi(g) = \operatorname{tr} \overline{\rho(g)}$ for all $g \in G$. χ is called the character of (ρ, V).

Definition

Suppose that G is a group, V is a finite-dimensional vector space and (ρ, V) is a representation of G. For each $g \in G$ consider the matrix representation of $\rho(g)$, denoted $\overline{\rho(g)}$, relative to some fixed basis in V. Define a function $\chi : G \to \mathbb{C}$ by $\chi(g) = \operatorname{tr} \overline{\rho(g)}$ for all $g \in G$. χ is called the character of (ρ, V).

It turns out that, if G is Abelian, the characters form a group. Call this the dual group of G and write it as $\tilde{G}$. Moreover $\tilde{G} \cong G$.

Definition

Suppose that G is a group, V is a finite-dimensional vector space and (ρ, V) is a representation of G. For each $g \in G$ consider the matrix representation of $\rho(g)$, denoted $\overline{\rho(g)}$, relative to some fixed basis in V. Define a function $\chi : G \to \mathbb{C}$ by $\chi(g) = \operatorname{tr} \overline{\rho(g)}$ for all $g \in G$. χ is called the character of (ρ, V).

It turns out that, if G is Abelian, the characters form a group. Call this the dual group of G and write it as $\tilde{G}$. Moreover $\tilde{G} \cong G$.

Definition

Let G be a group and χ be the character of some irreducible representation of G. Then define the Fourier transform of some measure p at χ as

$$\hat{p}(\chi) = \sum_{g \in G} p(g)\chi(g)$$

From now on we will take G to be Abelian. Then $G \cong \tilde{G}$ implies that the collection $(\hat{p}(\chi))_{\chi \in \tilde{G}}$ is precisely the spectrum of p viewed as a convolution operator.

From now on we will take G to be Abelian. Then $G \cong \tilde{G}$ implies that the collection $(\hat{p}(\chi))_{\chi \in \tilde{G}}$ is precisely the spectrum of p viewed as a convolution operator. Using this and

Theorem

ρ is an irreducible representation if and only if $d_\rho = 1$.

we get

$$|G| \sum_{g \in G} |p^{(t)}(g) - u(g)|^2 = \sum_{\chi \in \tilde{G}}^{*} |\hat{p}(\chi)|^{2t}$$

Consider $G = \mathbb{Z}_n$. We know that G is Abelian and so there are n irreducible representations given by

$$\rho_j(N) = (e^{2\pi ijN/n})$$

for any $N \in \mathbb{Z}_n$, $x \in \mathbb{C}^*$ and $0 \leq j < n$.

Consider $G = \mathbb{Z}_n$. We know that G is Abelian and so there are n irreducible representations given by

$$\rho_j(N) = (e^{2\pi ijN/n})$$

for any $N \in \mathbb{Z}_n$, $x \in \mathbb{C}^*$ and $0 \leq j < n$. Their characters are

$$\chi_j(N) = e^{2\pi ijN/n}$$

Consider $G = \mathbb{Z}_n$. We know that G is Abelian and so there are n irreducible representations given by

$$\rho_j(N) = (e^{2\pi ijN/n})$$

for any $N \in \mathbb{Z}_n$, $x \in \mathbb{C}^*$ and $0 \leq j < n$. Their characters are

$$\chi_j(N) = e^{2\pi ijN/n}$$

Consider the simple random walk where $p(+1) = p(-1) = 1/2$. Then

$$\begin{aligned}\hat{p}(\chi_j) &= \frac{1}{2}(\chi(+1) + \chi(-1)) \\ &= \frac{1}{2}\left(e^{t\pi ij/n} + e^{-t\pi ij/n}\right) \\ &= \cos(2\pi ij/n)\end{aligned}$$

Consider $G = \mathbb{Z}_n$. We know that G is Abelian and so there are n irreducible representations given by

$$\rho_j(N) = (e^{2\pi ijN/n})$$

for any $N \in \mathbb{Z}_n$, $x \in \mathbb{C}^*$ and $0 \leq j < n$. Their characters are

$$\chi_j(N) = e^{2\pi ijN/n}$$

Consider the simple random walk where $p(+1) = p(-1) = 1/2$. Then

$$\begin{aligned}\hat{p}(\chi_j) &= \frac{1}{2}(\chi(+1) + \chi(-1)) \\ &= \frac{1}{2}\left(e^{t\pi ij/n} + e^{-t\pi ij/n}\right) \\ &= \cos(2\pi ij/n)\end{aligned}$$

Thus

$$\|p^{(t)} - u\|_{\mathrm{TV}}^2 \asymp \sum_{j=1}^{n-1} |\cos(2\pi ij/n)|^{2t}$$